Relaxation Oscillations and chaotic motion in a system of nonlinear coupled oscillators

Conference Paper · July 2011

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Relaxation Oscillations and chaotic motion in a system of nonlinear coupled oscillators

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Summary. We study a three degrees of freedom autonomous system with damping and an essential nonlinear interaction between one of the linear oscillators and the small nonlinear attachment. We reduce our system to a non-autonomous second order nonlinear damped oscillator. After averaging out the fast frequencies we study our reduced system through the Slow Invariant Manifold (SIM) approach. With the help of the SIM we can identify the parameters in order to predict the asymptotic behavior of the orbits of our reduced system. This is accomplished with the application of Tikhonov’s theorem. We show that for a damping below a critical threshold there exists relaxation oscillations. From the numerical study of the reduced system we see numerical evidence of the existence of the relaxation oscillations and also of probably chaotic motion.

Introduction

There is a great interest in structures of linear systems with a nonlinear attachment [2, 3, 4, 9, 11, 12]. The nonlinear attachment has a small mass in comparison to the whole structure and the system is non conservative. A special property of the above systems is that the nonlinear attachment can act as a nonlinear energy sink (NES). In the case of external forcing various other complicated dynamical phenomena may appear [2, 3]. We will study such a system of 3 coupled oscillators

\[
\begin{align*}
\epsilon \ddot{y} + \epsilon \lambda (\dot{y} - \dot{x}_0) + C(y - x_0)^3 &= 0, \\
\dot{x}_0 + d(x_0 - x_1) &= \epsilon \lambda (\dot{y} - \dot{x}_0) + C(y - x_0)^3, \\
\dot{x}_1 + ax_1 + d(x_1 - x_0) &= 0.
\end{align*}
\]

(1)

where \( \epsilon \ll 1 \) is a small parameter which scales the coupling between the damping forces and the mass of the light attachment and where we expect to have energy transfer. Through singular transformations and reduction of the system we derive a non autonomous damped strongly nonlinear second order differential equation. With the use of the complexification-averaging technique (CX-A) [8] we obtain a system of two, first order, differential equations governed by slow time. The Slow Invariant Manifold (SIM) of the above system, found through multiple scale analysis or singularity analysis [1, 5, 6, 7, 10, 13], gives information about the asymptotic behavior of the orbits of the reduced system. The SIM of the system may have one or three branches depending on the slow time and the value of the damping parameter and due to these bifurcations the dynamical behavior of the system changes drastically. Tikhonov’s theorem guarantees that, when the branches of the SIM are isolated and stable, the orbits of the system tend to the stable ones as long as they exist. The application of Tikhonov’s theorem allowed us to classify the behavior of the orbits in comparison with the evolution of the SIM, that depends on the parameters of the system. The theoretical analysis give us the range of the parameters for every case and allow us to predict the long term behavior of the orbits of our reduced system.

The damping parameter, \( \lambda \), and the number of the branches of the SIM plays an essential role in the evolution of the system. The dynamics can be simple, as in the case where there is only one persisting branch, or complicated, when different kinds of bifurcations occur, resulting in different phenomena such as relaxation oscillations and orbit excitations.

System Reduction and Slow Invariant Manifold approach

The above system has a complicated structure so we reduce it to a strongly nonlinear, non homogeneous equation [11, 12]. We apply the linear singular transformation \( v = \epsilon^{-1/2} x_0 + \epsilon^{1/2} y, \quad w = \epsilon^{-1/2} x_0 - \epsilon^{-1/2} y \) and the system becomes

\[
\begin{align*}
\dot{w} + \lambda (1 + \epsilon) \dot{w} + C(1 + \epsilon) w^3 &= -d \frac{v + \epsilon w}{1 + \epsilon} + dx_1, \\
\dot{v} + \frac{d}{1 + \epsilon} v - dx_1 &= -\frac{\epsilon dw}{1 + \epsilon}, \\
\dot{x}_1 + (a + d)x_1 - \frac{d}{1 + \epsilon} v &= \frac{\epsilon dw}{1 + \epsilon}.
\end{align*}
\]

(2)

Assuming that initially the linear oscillators \( v = K_1 z_1 + K_2 z_2, \quad x_1 = z_1 + z_2, \) are at zero and the initial velocities are \( \dot{z}_1(0) = \omega_1 z_{10}, \quad \dot{z}_2(0) = \omega_2 z_{20}, \) we obtain the approximate reduced system

\[
\dot{w} + \lambda (1 + \epsilon) \dot{w} + C(1 + \epsilon) w^3 = A \sin \omega_1 t + B \sin \omega_2 t + O(\epsilon),
\]

(3)

where

\[
A = (d - \frac{dK_1}{1 + \epsilon}) z_{10},
\]

(4)
and

\[ B = (d - \frac{dK_2}{1 + \epsilon})z_{20}, \]

where \((K_1, 1)\) and \((K_2, 1)\) are the eigenvectors of second and third equations of (2). Since the initial system and the transformations contain the small parameter \(\epsilon\) that can be used for performing singularity analysis, the Slow Invariant Manifold (SIM) approach will be used to analyze its behavior. By complexification-Averaging (CX-A) methodology and by transforming the reduced equation to the amplitude \(N\) and the phase \(\eta\) we derive the equations

\[
\begin{align*}
N' + \frac{\lambda N}{2} &= -A \sin(\eta) + \frac{B}{2} \sin(\epsilon \hat{B} t - \eta), \\
N\eta' + \frac{N}{2} - 3\bar{C}N^3 - 8 &= -\frac{A}{2} \cos(\eta) - \frac{B}{2} \cos(\epsilon \hat{B} t - \eta).
\end{align*}
\]

We use the multiple scales analysis [12]

\[
N(t) = N(t_0, t_1, ...) = N_0(t_0, t_1, ...) + \epsilon N_1(t_0, t_1, ...) + O(\epsilon),
\]

\[
\eta(t) = \eta(t_0, t_1, ...) = \eta_0(t_0, t_1, ...) + \epsilon \eta_1(t_0, t_1, ...) + O(\epsilon),
\]

where \(t_0 = t\) and \(t_1 = \epsilon t\). By keeping \(O(1)\) terms we derive the equations

\[
\begin{align*}
\frac{\partial N_0}{\partial t_0} + \frac{\lambda N_0}{2} + \frac{A}{2} \sin(\eta_0) - \frac{B}{2} \sin(\epsilon \hat{B} t_0 - \eta_0) &= 0, \\
N_0 \frac{\partial \eta_0}{\partial t_0} + \frac{N_0}{2} - 3\bar{C}N_0^3 - 8 &= -\frac{A}{2} \cos(\eta_0) + \frac{B}{2} \cos(\epsilon \hat{B} t_0 - \eta_0).
\end{align*}
\]

To study the steady state dynamics of the above system, in terms of the fast time scale \(t_0\), we examine the limit of the dynamics as \(t_0 \to \infty\) and \(\frac{\partial N_0}{\partial t_0} = 0\). Then, from equations (8) we find

\[
\begin{align*}
\frac{\lambda N_0}{2} &= -A \sin(\eta_0) + \frac{B}{2} \sin(\epsilon \hat{B} t_0 - \eta_0), \\
\frac{N_0}{2} - 3\bar{C}N_0^3 - 8 &= -\frac{A}{2} \cos(\eta_0) + \frac{B}{2} \cos(\epsilon \hat{B} t_0 - \eta_0).
\end{align*}
\]

Manipulating expressions (9) we derive the steady state phase

\[
sin(\hat{\eta}_0) = -\frac{\hat{B} \sin(\epsilon \hat{B} t_0) \left(\frac{3\bar{C}N_0^3}{4} - \hat{N}_0\right) - \lambda \hat{N}_0 (\hat{A} + \hat{B} \cos(\epsilon \hat{B} t_0))}{A^2 + 2\hat{A}\hat{B}\cos(\epsilon \hat{B} t_0) + \hat{B}^2},
\]

with the steady state amplitude given by

\[
N_0^6 - \frac{8}{3\bar{C}} N_0^4 + \frac{16(\hat{A}^2 + 1)}{9\bar{C}^2} N_0^2 = \frac{16}{9\bar{C}^2} (A^2 + 2\hat{A}\hat{B}\cos(\epsilon \hat{B} t_0) + \hat{B}^2).
\]

Equations (10) and (11) represent the slow invariant manifold (SIM) of the dynamics of (3). The above analysis with multiple scales is equivalent to singularity analysis by taking the slow time \(t \to \epsilon \hat{B} t\). Then system (6) becomes

\[
\begin{align*}
\epsilon N' &= f(N, n, t), \\
\epsilon Nn' &= g(N, n, t)
\end{align*}
\]

and at the singular limit \((\epsilon = 0)\) we have \(f = g = 0\) which is exactly the SIM of the system. Tichonov’s theorem [10, 13] guarantees that, when the roots of \(f = g = 0\) are isolated and at the same time they are stable solutions of

\[
\begin{align*}
\frac{dN}{d\tau} &= f(N, n, t), \\
N\frac{dn}{d\tau} &= g(N, n, t)
\end{align*}
\]

with \(t\) considered as a parameter, then \(N(t) \to \hat{N}_0(t), \eta(t) \to \hat{\eta}_0(t)\) as \(\epsilon \to 0\). In order to find when the SIM is stable, according to Tichonov’s theorem, we find the linear stability of the roots of (9) when they are isolated.
Study of the SIM, Relaxation Oscillations

It can be seen from (11) and from the study of the stability matrix of the steady state solutions of (13) that the term \( \Sigma = \hat{A}^2 + 2AB\cos(\epsilon \hat{B}t) + \hat{B}^2 \) is important for the appearance and disappearance of the branches of the SIM and their stability. When (11) possesses a single real positive root (figure 1.a) the SIM has one periodic branch which attracts the neighbouring orbits, as Tikhonov’s theorem guarantees. In the case that (11) possesses 3 real positive roots (figure 1.b) bifurcations occur that lead to the change of stability of the branches of the SIM and the existence of relaxation oscillations.

We show analytically that in order to have 3 real positive roots it must hold that \( \hat{\lambda} < \frac{1}{\sqrt{3}} \). We also find the period of time for which the SIM have three branches. For the time interval that \( \Sigma > \frac{8}{81C}((1 + 9\hat{\lambda}^2) + (1 - 3\hat{\lambda}^2)^\frac{3}{2}) \) the upper branch exists and it is stable. For the period of time that \( \frac{8}{81C}((1 + 9\hat{\lambda}^2) - (1 - 3\hat{\lambda}^2)^\frac{3}{2}) < \Sigma < \frac{8}{81C}((1 + 9\hat{\lambda}^2) + (1 - 3\hat{\lambda}^2)^\frac{3}{2}) \) the three branches coexist the middle one is unstable while the lower and the upper branches are stable. Finally when \( \Sigma < \frac{8}{81C}((1 + 9\hat{\lambda}^2) - (1 - 3\hat{\lambda}^2)^\frac{3}{2}) \) the SIM has only one branch, the lower, and is stable.

We perform numerical simulations of system (6) from which we verify our analytical result for \( \hat{A}, \hat{B} \) and \( \hat{\lambda} \). In the case of relaxation oscillations, bifurcations occur between all the branches. For a certain period of time only the upper branch exists. Then, after a saddle-node bifurcation, two other branches appear, the one stable and the other unstable. Then the upper branch and the unstable one coalesce through a saddle-node bifurcation. Therefore, the orbits of (6) that are attracted to the upper branch of the SIM, suddenly jump to the lower branch producing relaxation oscillations (figure 2). Then the upper branch appear and the lower one disappear again through a saddle node bifurcation. This behavior is repeated periodically. These relaxation oscillations indicates a transient energy transfer in our reduced system.

For small values of \( \lambda \) and when we have a crossing of one of the stable branch of the SIM with the unstable branch, but there is no coalescence, the orbits of (6) may oscillate rapidly, while approaching the stable branch of the SIM, and follow it until the time where we have the crossing (but no disappearance) of the stable and unstable branch of the SIM. Then the orbit is excited and the phenomenon it repeated (figure 3). This behavior may relate to transverse homoclinic intersection of the stable and unstable manifolds of the saddle type branch of the SIM and may lead to chaotic motion.

Further investigation of the above phenomenon is needed both numerically and analytically in the future.
Conclusions

The study of a three degrees of freedom autonomous system with an essential nonlinear interaction with damping is performed through the Slow Invariant Manifold (SIM) of a reduced non-autonomous second order differential equation. Depending on the parameters of the system we have shown, through dynamical analysis and Tikhonov’s theorem, that the SIM can either have one branch that is stable, or three branches, two of them stable.

For large values of the damping parameter, the structure of the SIM is simple and the orbit are attracted to it. The interplay between the stable and the unstable branches of the SIM produces interesting phenomena such as relaxation oscillations and excitation of the orbits.

Relaxation oscillations occur when we have bifurcations of all three branches and are related to transient energy transfer of the reduced system (3). Although we have not shown it analytically, from the numerical study, we expect that the basin of attraction of the stable branch of the SIM (in the period of time that we have only one stable branch) is the whole state space. The excitation of the orbits may relate to transverse homoclinic intersection of the stable and unstable manifolds of the saddle type branch of the SIM. Further investigation of the above phenomena is needed both numerically and analytically in the future.

References